

RATIONAL TORSION POINTS ON JACOBIANS OF SHIMURA CURVES

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ABSTRACT. Let p and q be distinct primes. There is the Shimura curve \mathcal{X}^{pq} associated to the indefinite quaternion algebra of discriminant pq over \mathbb{Q} . Let J^{pq} be the Jacobian variety of \mathcal{X}^{pq} , which is an abelian variety over \mathbb{Q} . For an odd prime ℓ , we provide sufficient conditions for the non-existence of rational points of order ℓ on J^{pq} . As an application, we find some non-trivial subgroups of the kernel of an isogeny from the new quotient $J_0(pq)^{\text{new}}$ of $J_0(pq)$ to J^{pq} .

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1. INTRODUCTION

Let p and q be distinct primes. Consider the modular curves $X_0(p)$ and $X_0(pq)$ over \mathbb{Q} ; and their Jacobians $J_0(p)$ and $J_0(pq)$ over \mathbb{Q} . By Mordell-Weil theorem, the rational points on $J_0(p)$ and $J_0(pq)$ are finitely generated abelian groups, and hence

$$J_0(p)(\mathbb{Q}) \simeq \mathbb{Z}^a \oplus J_0(p)(\mathbb{Q})_{\text{tor}} \quad \text{and} \quad J_0(pq)(\mathbb{Q}) \simeq \mathbb{Z}^b \oplus J_0(pq)(\mathbb{Q})_{\text{tor}},$$

where $J_0(p)(\mathbb{Q})_{\text{tor}}$ and $J_0(pq)(\mathbb{Q})_{\text{tor}}$ are finite abelian groups.

In early 1970s Ogg [11] conjectured that the group $J_0(p)(\mathbb{Q})_{\text{tor}}$ is generated by the cuspidal divisor $[(0) - (\infty)]$. In his landmark paper [10], Mazur proved Ogg's conjecture. To do this, he studied submodules of $J_0(p)$ annihilated by the Eisenstein ideal of the Hecke ring of level p . A natural generalization is as follows.

Conjecture 1.1 (Generalized Ogg's conjecture). *All rational torsion points on $J_0(pq)$ are cuspidal, i.e.,*

$$J_0(pq)(\mathbb{Q})_{\text{tor}} = \mathcal{C}(pq),$$

where $\mathcal{C}(pq)$ is the cuspidal group of $J_0(pq)$.

Our present knowledge is insufficient to prove the above conjecture completely. As Mazur pointed out [10, p. 34], control of the 2-torsion part of $J_0(pq)(\mathbb{Q})_{\text{tor}}$ is very difficult. Note that some of this conjecture is now proved by Ohta [13] and by the author [26].

In this paper, instead of studying the above conjecture, we consider an abelian variety J^{pq} , which is isogenous to the new quotient $J_0(pq)^{\text{new}}$ of $J_0(pq)$. More specifically, let \mathcal{X}^{pq} be the Shimura curve associated to the indefinite quaternion algebra over \mathbb{Q} of discriminant pq with trivial level structure. Let J^{pq} be the Jacobian variety of \mathcal{X}^{pq} , which is an abelian variety over \mathbb{Q} of dimension $g(\mathcal{X}^{pq})$, the genus of \mathcal{X}^{pq} . From now on, we always assume that $g(\mathcal{X}^{pq}) \neq 0$. Then, we prove the following theorem.

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Theorem 1.2. *For a prime $\ell \geq 5$, the Jacobian J^{pq} does not have rational points of order ℓ unless one of the following holds:*

- $p \equiv q \equiv 1 \pmod{\ell}$;
- $p \equiv 1 \pmod{\ell}$ and $q^{\frac{p-1}{\ell}} \equiv 1 \pmod{p}$;
- $q \equiv 1 \pmod{\ell}$ and $p^{\frac{q-1}{\ell}} \equiv 1 \pmod{q}$.

Furthermore, the Jacobian J^{pq} does not have rational points of order 3 if $(p-1)(q-1)$ is not divisible by 3.

As an application of this theorem, we get information about the kernel of an isogeny between $J_0(pq)^{\text{new}}$ and J^{pq} . (The existence of this isogeny is due to Ribet [16, Théorème 2].) More precisely, let $\Psi(pq)$ denote such an isogeny over \mathbb{Q} , and let $K(pq)$ denote the kernel of $\Psi(pq)$. By the careful study of bad reduction of Shimura curves, Ogg [12] conjectured that the image of some cuspidal divisors in $J_0(pq)$ belongs to $K(pq)$. In the case of low genus Shimura curves, this conjecture was proved by González and Molina [7]. More precisely, they found an equation of \mathcal{X}^{pq} in the case where $g(\mathcal{X}^{pq}) \leq 3$ and computed $K(pq)$ (for chosen $\Psi(pq)$) by the consideration of bad reduction of \mathcal{X}^{pq} . Instead of finding an explicit equation of \mathcal{X}^{pq} and computing $K(pq)$, we prove that $K(pq)$ always contains $\pi(\mathcal{C}_\ell(pq))$ if ℓ satisfies certain conditions, where $\mathcal{C}_\ell(pq)$ is the ℓ -primary subgroup of $\mathcal{C}(pq)$ and π is the quotient map from $J_0(pq)$ to $J_0(pq)^{\text{new}}$.

Theorem 1.3. *Let ℓ^m and ℓ^n be the exact powers of ℓ dividing $p+1$ and $q+1$, respectively. If $\ell \geq 5$ and all the following conditions hold, then $K(pq)$ contains $\pi(\mathcal{C}_\ell(pq))$, and the latter is isomorphic to $\mathbb{Z}/\ell^m\mathbb{Z} \oplus \mathbb{Z}/\ell^n\mathbb{Z}$:*

- ℓ does not divide $(p-1, q-1)$;
- if $p \equiv 1 \pmod{\ell}$, then $q^{\frac{p-1}{\ell}} \not\equiv 1 \pmod{p}$;
- if $q \equiv 1 \pmod{\ell}$, then $p^{\frac{q-1}{\ell}} \not\equiv 1 \pmod{q}$.

If $\ell = 3$ and $(p-1)(q-1)$ is not divisible by 3, then $K(pq)$ contains $\pi(\mathcal{C}_3(pq))$, and the latter is isomorphic to $\mathbb{Z}/3^\alpha\mathbb{Z} \oplus \mathbb{Z}/3^\beta\mathbb{Z}$, where $\alpha = \max\{0, m-1\}$ and $\beta = \max\{0, n-1\}$.

The organization of this article is as follows. In §2, we discuss all possible new Eisenstein maximal ideals of level pq . In §3, we give certain criteria on primes p and q for an Eisenstein ideal discussed in the previous section to be maximal. In §4, we study the structures of the kernels of Eisenstein maximal ideals on Jacobians. In §5, we deduce Theorem 1.2 from the above results. Finally, we prove Theorem 1.3 in §6.

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1.1. Notation. Let B be a quaternion algebra over \mathbb{Q} of discriminant D such that $\phi : B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R})$. Let \mathcal{O} be an Eichler order of level N of B and let $\mathcal{O}^{\times,1}$ be the set of (reduced) norm 1 elements in \mathcal{O} . We define $\Gamma_0^D(N) := \phi(\mathcal{O}^{\times,1})$. Let $X_0^D(N)$ be the Shimura curve over \mathbb{Q} associated to B with $\Gamma_0^D(N)$ level structure and let $J_0^D(N) := \text{Pic}^0(X_0^D(N))$ be its Jacobian variety. If $D = 1$, then $X_0(N) = X_0^1(N)$ denotes the modular curve for $\Gamma_0(N)$ and $J_0(N) = J_0^1(N)$ denotes its Jacobian variety. If $D \neq 1$, then $X_0^D(N)(\mathbb{C}) \simeq \Gamma_0^D(N) \backslash \mathbb{H}$, where \mathbb{H} is the complex upper half plane.

For an integer $n \geq 1$, there is a Hecke operator T_n acting on $J_0^D(N)$. We denote by $\mathbb{T}^D(N)$ the \mathbb{Z} -subalgebra of the endomorphism ring of $J_0^D(N)$ generated by all T_n . In the case where $D = 1$ (resp. $N = 1$), we simply denote by $\mathbb{T}(N)$ (resp. \mathbb{T}^D) the Hecke ring $\mathbb{T}^1(N)$ (resp. $\mathbb{T}^D(1)$). If p divides DN , we often denote by U_p the p^{th} Hecke operator T_p on $J_0^D(N)$. For a prime p dividing N , there is also an Atkin-Lehner involution w_p on $J_0^D(N)$. For a maximal ideal \mathfrak{m} of a Hecke ring \mathbb{T} , we denote by $\mathbb{T}_{\mathfrak{m}}$ the completion of \mathbb{T} at \mathfrak{m} , i.e.,

$$\mathbb{T}_{\mathfrak{m}} := \varprojlim_n \mathbb{T}/\mathfrak{m}^n.$$

2. EISENSTEIN IDEALS IN \mathbb{T}^{pq}

From now on, we fix distinct primes p and q ; and ℓ denotes an odd prime. Let $\mathbb{T} := \mathbb{T}^{pq}$ and $I_0 := (T_r - r - 1 : \text{for primes } r \nmid pq) \subset \mathbb{T}$.

Lemma 2.1. *We have $U_p^2 = U_q^2 = 1 \in \mathbb{T}$.*

Proof. Let w_p and w_q be Atkin-Lehner involutions on $J_0(pq)$. Then $U_p + w_p = 0$ on the space of newforms (cf. [19, Proposition 3.7]). Because $\Psi(pq)$ is Hecke-equivariant (cf. [19, §4]), U_p and U_q are also involutions on J^{pq} . \square

Definition 2.2. We define Eisenstein ideals containing I_0 as follows:

$$\begin{aligned} I_1 &:= (U_p - 1, U_q - 1, I_0), & I_2 &:= (U_p + 1, U_q + 1, I_0), \\ I_3 &:= (U_p - 1, U_q + 1, I_0) & \text{and} & \quad I_4 := (U_p + 1, U_q - 1, I_0). \end{aligned}$$

Moreover, we set $\mathfrak{m}_i := (\ell, I_i)$. They are all possible Eisenstein maximal ideals in \mathbb{T} by the above lemma.

Let $\mathbb{T}_\ell := \mathbb{T} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. Then it is a semi-local ring and we have

$$\mathbb{T}_\ell = \prod_{\ell \in \mathfrak{m} \text{ maximal}} \mathbb{T}_\mathfrak{m}.$$

Using the above description of Eisenstein maximal ideals, we prove the following.

Proposition 2.3. *The quotient \mathbb{T}_ℓ/I_0 decomposes as follows:*

$$\mathbb{T}_\ell/I_0 = \prod_{i=1}^4 \mathbb{T}_{\mathfrak{m}_i}/I_0 \simeq \prod_{i=1}^4 \mathbb{T}_\ell/I_i.$$

Proof. It suffices to prove that $\mathbb{T}_\ell/I \simeq \mathbb{T}_\mathfrak{m}/I_0$, where $I := I_i$ and $\mathfrak{m} := (\ell, I)$. We discuss the case where $i = 1$ and other cases are basically the same.

If \mathfrak{m} is not maximal, then $\mathbb{T}_\mathfrak{m} = 0 = \mathbb{T}_\ell/I$. Therefore we may assume that \mathfrak{m} is maximal. Since ℓ is odd and $U_p - 1 \in \mathfrak{m}$, we have $U_p + 1 \notin \mathfrak{m}$. In other words, $U_p + 1$ is a unit in $\mathbb{T}_\mathfrak{m}$. By Lemma 2.1, we have $U_p - 1 = 0$ in $\mathbb{T}_\mathfrak{m}$ and hence $U_p - 1 \in I_0$. Similarly we have $U_q - 1 \in I_0$. Therefore $\mathbb{T}_\mathfrak{m}/I_0 = \mathbb{T}_\mathfrak{m}/I$. Since the index of I in \mathbb{T} is finite (cf. [23, Lemma 3.1]), there is an integer n such that $\mathfrak{m}^n \subseteq I$. Thus, we have $(\mathbb{T}_\ell/\mathfrak{m}^n)/I = \mathbb{T}_\ell/I$ and hence $\mathbb{T}_\mathfrak{m}/I \simeq \mathbb{T}_\ell/I$. \square

3. CRITERIA FOR \mathfrak{m} TO BE MAXIMAL

In this section, we discuss certain conditions on the primes p and q for which \mathfrak{m}_i is maximal. By the Jacquet-Langlands correspondence, it suffices to show that \mathfrak{m}_i is new maximal in $\mathbb{T}(pq)$ under the given assumption.

3.1. Maximality of \mathfrak{m}_1 .

Theorem 3.1. *The ideal \mathfrak{m}_1 is maximal in \mathbb{T} if and only if one of the following holds:*

- $p \equiv q \equiv 1 \pmod{\ell}$;
- ℓ divides the numerator of $\frac{p-1}{3}$ and $q^{\frac{p-1}{\ell}} \equiv 1 \pmod{p}$;
- ℓ divides the numerator of $\frac{q-1}{3}$ and $p^{\frac{q-1}{\ell}} \equiv 1 \pmod{q}$.

Proof. Let $\mathfrak{m} := \mathfrak{m}_1$ and $I := I_1$.

If $\ell \geq 5$ and ℓ does not divide pq , this is proved by Ribet [24, Theorem 2.4] (The proof of this theorem is given in §4 of *op. cit.*)

Since the index of I in $\mathbb{T}(pq)$ is equal to the numerator of $\frac{(p-1)(q-1)}{3}$ up to powers of 2 [25, Theorem 5.1], we assume that ℓ divides the numerator of $\frac{(p-1)(q-1)}{3}$, and hence \mathfrak{m} is maximal in $\mathbb{T}(pq)$.

Let $\ell = 3$ and let λ be the ideal of $\mathbb{T}(p)$ corresponding to \mathfrak{m} . By Mazur [10], λ is maximal if and only if $p \equiv 1 \pmod{9}$. First, we assume that $p \equiv 1 \pmod{9}$. Let $R := \mathbb{T}(p)_\lambda$ be the completion of $\mathbb{T}(p)$ at λ , and let I be the Eisenstein ideal of $\mathbb{T}(p)$. Then, $IR \neq (T_q - q - 1)R$ if and only if either $q \equiv 1 \pmod{3}$ or $q^{\frac{p-1}{3}} \equiv 1 \pmod{p}$. By the same argument as in the proof of [24, Theorem 2.4], \mathfrak{m} is new maximal if and only if $IR \neq (T_q - q - 1)R$. Therefore by symmetry, the result follows unless $p - 1$ and $q - 1$ are exactly divisible by 3. Next, we assume that $p - 1$ and $q - 1$ are exactly divisible by 3. Then \mathfrak{m} is new because it is neither p -old nor q -old.

If $\ell \geq 5$, the same method as above works and the result follows directly. \square

Remark 3.2. In the above proof, we don't need to assume that ℓ does not divide pq . Note that one direction in the proof of [24, Theorem 2.4] relies on the saturation property of $\mathbb{T}(pq)$ in $\text{End}(J_0(pq))$ locally at \mathfrak{m} . If either $\ell = p$ or $\ell = q$, this property follows from the second case of [24, Theorem 3.3] because $T_\ell \equiv 1 \pmod{\mathfrak{m}}$ (cf. [20, Lemma 1.1] or [24, Remark 3.5]). The other direction follows by the same argument as in the proof of [24, Theorem 2.4] without further difficulties.

3.2. Maximality of \mathfrak{m}_2 .

Theorem 3.3. *The ideal \mathfrak{m}_2 cannot be maximal.*

Proof. For an Eisenstein maximal ideal \mathfrak{m} , we have $T_\ell \equiv 1 \pmod{\mathfrak{m}}$. Therefore \mathfrak{m}_2 is not maximal if either $\ell = p$ or $\ell = q$ because ℓ is odd. Thus, we assume that ℓ does not divide pq and \mathfrak{m}_2 is maximal. By [24, Theorem 1.2.(3)], we have $p \equiv q \equiv -1 \pmod{\ell}$ and $\mathfrak{m}_2 = (\ell, U_p - p, U_q - q, I_0)$. By [25, Proposition 5.5], \mathfrak{m}_2 cannot be maximal, which is a contradiction. Therefore the result follows. \square

3.3. Maximality of \mathfrak{m}_3 and \mathfrak{m}_4 .

Theorem 3.4. *The ideal \mathfrak{m}_3 is maximal if and only if ℓ divides the numerator of $\frac{q+1}{(3, p(p+1))}$. By symmetry, the ideal \mathfrak{m}_4 is maximal if and only if ℓ divides the numerator of $\frac{p+1}{(3, q(q+1))}$.*

Proof. Let $\mathfrak{m} := \mathfrak{m}_3$ and $I := I_3$.

If $\ell \geq 5$ and ℓ does not divide pq , this is proved by Ribet [24, Theorem 1.4(2)]. (The proof of this theorem is given in §4 of *op. cit.*)

Let n be the numerator of $\frac{q+1}{\gcd(3, p(p+1))}$. Since \mathbb{T} is a quotient of $\mathbb{T}(pq)$ and the index of I in $\mathbb{T}(pq)$ is equal to n up to powers of 2 [23, Theorem 3.4], \mathfrak{m} is not maximal if ℓ does not divide n . Conversely, if ℓ divides n then $S_q[\mathfrak{m}] \neq 0$ by Proposition 4.4 below, where S_q is the Skorobogatov subgroup of J^{pq} from the level structure at q . Therefore \mathfrak{m} is maximal. \square

4. THE STRUCTURE OF $J^{pq}[\mathfrak{m}]$

Let $J := J^{pq}$. In this section, we discuss the structure of $J[\mathfrak{m}]$, where $\mathfrak{m} = \mathfrak{m}_i$ for $1 \leq i \leq 4$. If \mathfrak{m}_i is not maximal, then $J[\mathfrak{m}] = 0$. Therefore it suffices to study $J[\mathfrak{m}_1]$ and $J[\mathfrak{m}_3]$ by symmetry.

4.1. Multiplicity one for Jacobians of Shimura curves. In this subsection, we prove multiplicity one result as follows.

Theorem 4.1. *Assume that $\mathfrak{m} = \mathfrak{m}_3$ is maximal. If $\ell = 3$, we further assume that 3 does not divide $(p-1)(q-1)$. Then, $J[\mathfrak{m}]$ is a non-trivial extension of $\mathbb{Z}/\ell\mathbb{Z}$ by μ_ℓ . Moreover, $J[\mathfrak{m}]$ is ramified at p but is unramified at q . Therefore we have $\mathbb{Z}/\ell\mathbb{Z} \not\subseteq J[\mathfrak{m}]$.*

When $\mathfrak{m} = \mathfrak{m}_1$ is maximal, the structure of $J[\mathfrak{m}]$ is more complicated than one of $J[\mathfrak{m}_3]$. However, if one of $p-1$ and $q-1$ is not divisible by ℓ , then we have the similar result as above. Note that the theorem below is not used in the proof of our main theorem.

Theorem 4.2. *Assume that $\mathfrak{m} = \mathfrak{m}_1$ is maximal. Assume further $\ell \geq 5$ and $q \not\equiv 1 \pmod{\ell}$. Then, $J[\mathfrak{m}]$ is of dimension 2 and is ramified at p .*

Proof of Theorem 4.1. Let $\mathfrak{m} = \mathfrak{m}_3$. By Theorem 3.4, \mathfrak{m} is maximal if and only if ℓ divides the numerator of $\frac{q+1}{\gcd(3, p(p+1))}$. Hence in particular, we assume that $q \not\equiv 1 \pmod{\ell}$.

For the Jacobian $J_0^D(N)$ with N square-free, we denote by $J_0^D(N)_{/\mathbb{F}_p}$ be the special fiber of the Néron model of $J_0^D(N)$ over \mathbb{F}_p . If p is a divisor of N (resp. of D), then it is given by a Deligne-Rapoport model [1, 4] (resp. a Cerednik-Drinfeld model [2, 5]) and the theory of Raynaud [14]. We denote by $\Phi_p(J_0^D(N))$ (resp. $X_p(J_0^D(N))$) the component group of $J_0^D(N)_{/\mathbb{F}_p}$ (resp. the character group of $J_0^D(N)_{/\mathbb{F}_p}$).

We shall carry out a proof by several steps.

- Step 1 : We show that $\Phi_p(J)[\mathfrak{m}] = 0$ as follows.

By Ribet [19, Theorem 4.3], there is a Hecke-equivariant exact sequence:

$$0 \longrightarrow K \longrightarrow (X \oplus X)/\delta_p(X \oplus X) \longrightarrow \Phi_p(J) \longrightarrow C \longrightarrow 0,$$

where $X := X_q(J_0(q))$ and $\delta_p = \begin{pmatrix} p+1 & T_p \\ T_p & p+1 \end{pmatrix}$; and K (resp. C) is the kernel (resp. the cokernel) of the map

$$\gamma_p : \Phi_q(J_0(q)) \times \Phi_q(J_0(q)) \rightarrow \Phi_q(J_0(pq))$$

induced by the degeneracy map $\gamma_p : J_0(q) \times J_0(q) \rightarrow J_0(pq)$. Since $q \not\equiv 1 \pmod{\ell}$, there is no Eisenstein ideal of level q . Therefore the first and second terms of the above exact sequence have no support at \mathfrak{m} .

- If $\ell \geq 5$, then $C[\mathfrak{m}] = 0$ by [24, Proposition A.5] and [24, Corollary A.6]. Therefore $\Phi_p(J)[\mathfrak{m}] = 0$.
- If $\ell = 3$ and $q > 3$, then the 3-primary part of $\Phi_q(J_0(pq))$ is cyclic by [6, §4.4.1] because $p \not\equiv 1 \pmod{3}$. Since U_q acts as 1 on it (cf. [24, Proposition A.2]), we have $C[\mathfrak{m}] = 0$ and hence $\Phi_p(J)[\mathfrak{m}] = 0$.
- If $\ell = 3$ and $q = 2$, we have $\Phi_p(J)[\mathfrak{m}] = 0$ by the table in [12, p. 210] because $p \equiv -1 \pmod{3}$.

- Step 2 : We show that $\mathbb{T}_{\mathfrak{m}}$ is Gorenstein as follows.

Let $Y = X_p(J)$ and $L = X_q(J_0(pq))$. By Ribet [19], there is a Hecke-equivariant exact sequence:

$$0 \longrightarrow Y \longrightarrow L \longrightarrow X \oplus X \longrightarrow 0.$$

By taking completions at \mathfrak{m} , we have $Y_{\mathfrak{m}} \simeq L_{\mathfrak{m}}$. By [23, Theorem 4.5.(4)], the dimension of $J_0(pq)[\mathfrak{m}]$ is either 2 or 3. However, the dimension of $L/\mathfrak{m}L$ is 1 in both cases. Therefore, $Y/\mathfrak{m}Y$ is of dimension 1 as well and $Y_{\mathfrak{m}}$ is free of rank 1 over $\mathbb{T}_{\mathfrak{m}}$. Moreover by the monodromy exact sequence, we have a Hecke-equivariant exact sequence:

$$0 \longrightarrow Y \longrightarrow \text{Hom}(Y, \mathbb{Z}) \longrightarrow \Phi_p(J) \longrightarrow 0.$$

Since $\Phi_p(J)[\mathfrak{m}] = 0$, we have $Y_{\mathfrak{m}} \simeq \text{Hom}(Y_{\mathfrak{m}}, \mathbb{Z}_{\ell})$. In other words, $Y_{\mathfrak{m}}$ is a free self-dual $\mathbb{T}_{\mathfrak{m}}$ -module of rank 1, and hence $\mathbb{T}_{\mathfrak{m}}$ is Gorenstein.

- Step 3 : We show that $J[\mathfrak{m}]$ is of dimension 2 as follows.

By Grothendieck [8], there is an exact sequence:

$$0 \longrightarrow \text{Hom}(Y/\ell^n Y, \mu_{\ell^n}) \longrightarrow J[\ell^n] \longrightarrow Y/\ell^n Y \longrightarrow 0.$$

(For details, see [15, §3.3].) By taking projective limits, we have

$$0 \longrightarrow \text{Hom}(Y_{\ell}, \mathbb{Z}_{\ell}(1)) \longrightarrow \text{Ta}_{\ell} J \longrightarrow Y_{\ell} \longrightarrow 0,$$

where $\text{Ta}_{\ell} J$ is the ℓ -adic Tate module of J and $\mathbb{Z}_{\ell}(1)$ is the Tate twist of \mathbb{Z}_{ℓ} . Since $\mathbb{T}_{\mathfrak{m}}$ is a direct factor of \mathbb{T}_{ℓ} , we have

$$0 \longrightarrow \text{Hom}(Y_{\mathfrak{m}}, \mathbb{Z}_{\ell}(1)) \longrightarrow \text{Ta}_{\mathfrak{m}} J \longrightarrow Y_{\mathfrak{m}} \longrightarrow 0.$$

Since $Y_{\mathfrak{m}}$ is a free self-dual $\mathbb{T}_{\mathfrak{m}}$ -module of rank 1, $\text{Ta}_{\mathfrak{m}} J$ is a free $\mathbb{T}_{\mathfrak{m}}$ -module of rank 2. Therefore $J[\mathfrak{m}]$ is of dimension 2.

- Step 4 : We show that $J[\mathfrak{m}]$ is ramified at p as follows.

Let I_p be an inertia subgroup of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ at p . Then by Serre-Tate [21], we have $J[\mathfrak{m}]^{I_p} \simeq J/\mathbb{F}_p[\mathfrak{m}]$. Since $\Phi_p(J)[\mathfrak{m}] = 0$ and $J^0[\mathfrak{m}] = \text{Hom}(Y/\mathfrak{m}Y, \mu_{\ell})$ is of dimension 1, $J[\mathfrak{m}]^{I_p} \simeq J^0[\mathfrak{m}]$ is of dimension 1 as well, where J^0 is the identity component of J/\mathbb{F}_p . Therefore $J[\mathfrak{m}]$ is ramified at p .

- Step 5 : We show that $J[\mathfrak{m}]$ contains μ_{ℓ} as follows.

By Proposition 4.4 below, our assumption on \mathfrak{m} implies $\mu_{\ell} \simeq S_q[\mathfrak{m}] \subseteq J[\mathfrak{m}]$, where S_q is the Skrobogotov subgroup of J from the level structure at q .

- Step 6 : We show that $J[\mathfrak{m}]$ is a non-trivial extension of $\mathbb{Z}/\ell\mathbb{Z}$ by μ_{ℓ} as follows.

Since all Jordan-Hölder factors of $J[\mathfrak{m}]$ are either μ_{ℓ} or $\mathbb{Z}/\ell\mathbb{Z}$ (cf. [10, Proposition 14.1]), the quotient $J[\mathfrak{m}]/\mu_{\ell}$ is isomorphic to either $\mathbb{Z}/\ell\mathbb{Z}$ or μ_{ℓ} . If $J[\mathfrak{m}]/\mu_{\ell} \simeq \mu_{\ell}$, then $J[\mathfrak{m}^{\infty}]$ is a multiplicative \mathfrak{m} -divisible

module, which is a contradiction. Therefore $J[\mathfrak{m}]$ is an extension of $\mathbb{Z}/\ell\mathbb{Z}$ by μ_ℓ . Since $J[\mathfrak{m}]$ is ramified at p , we have $\mathbb{Z}/\ell\mathbb{Z} \not\subseteq J[\mathfrak{m}]$.

- Step 7 : We finish the proof by showing that $J[\mathfrak{m}]$ is unramified at q as follows.

Let Frob_q be the Frobenius endomorphism in characteristic q . Then, Frob_q acts by qU_q on the torus T of J/\mathbb{F}_q (cf. [9, Theorem 3.1], [18]). Since $U_q \equiv -1 \pmod{\mathfrak{m}}$, $T[\mathfrak{m}]$ cannot contain $\mu_\ell \subseteq J[\mathfrak{m}]^{I_q}$. Since \mathbb{T} acts faithfully on $X_q(J)$ and \mathfrak{m} is maximal, $T[\mathfrak{m}]$ is at least of dimension 1. Therefore $J[\mathfrak{m}]^{I_q} \simeq J/\mathbb{F}_q[\mathfrak{m}]$ is at least of dimension 2. Thus, $J[\mathfrak{m}]$ is unramified at q . \square

Remark 4.3. Most of the above proof was given by Ribet in [24, Appendix B] with the assumption that $p \not\equiv 1 \pmod{\ell}$ and $\ell \geq 5$. However we duplicate the proof here to point out where our assumption plays a role.

Proof of Theorem 4.2. Since $q \not\equiv 1 \pmod{\ell}$, \mathfrak{m} is not p -old by Mazur [10]. Moreover we have $C[\mathfrak{m}] = 0$ as in Step 1 of the above proof because U_q acts by q on $C[\ell]$. Therefore the argument in Step 1 works in this case as well. With our assumption on q , the arguments in Steps 2–4 are also valid as above, and hence the result follows. \square

4.2. The Skorobogatov subgroups of J . In this subsection, we discuss a subgroup of $J[\mathfrak{m}_3]$, which is the ℓ -torsion subgroup of the Skorobogatov subgroup S_q from the level structure at q . In [24, Appendix C], we studied the actions of the Hecke operators on S_q and computed its order up to products of powers of 2 and 3. Since we include the discussion with $\ell = 3$, we compute the ℓ -torsion subgroup on S_q for any odd prime ℓ .

Proposition 4.4. *We have $S_q[\ell] \neq 0$ if and only if ℓ divides the numerator of $\frac{q+1}{\gcd(3, p(p+1))}$. If $S_q[\ell] \neq 0$, then we have $S_q[\ell] = S_q[\mathfrak{m}_3] \simeq \mu_\ell$.*

Proof. Since the order of S_q is equal to $\frac{q+1}{\epsilon(q)}$ (up to powers of 2), the first statement follows by the definition of $\epsilon(q)$ in [22, p. 781]. Since S_q is the Cartier dual of the constant cyclic group scheme (cf. *loc. cit.*), $S_q[\ell]$ is isomorphic to μ_ℓ if it is not zero. Therefore we have $S_q[\ell] = S_q[\mathfrak{m}_3] \simeq \mu_\ell$ by [24, Proposition C.2] if $S_q[\ell] \neq 0$. \square

5. NON-EXISTENCE OF RATIONAL POINTS OF ORDER ℓ ON J^{pq}

In this section, we prove our main theorem.

Theorem 5.1. *For a prime $\ell \geq 5$, the Jacobian J^{pq} does not have rational points of order ℓ unless one of the following holds:*

- $p \equiv q \equiv 1 \pmod{\ell}$;
- $p \equiv 1 \pmod{\ell}$ and $q^{\frac{p-1}{\ell}} \equiv 1 \pmod{p}$;
- $q \equiv 1 \pmod{\ell}$ and $p^{\frac{q-1}{\ell}} \equiv 1 \pmod{q}$.

Furthermore, the Jacobian J^{pq} does not have rational points of order 3 if $(p-1)(q-1)$ is not divisible by 3.

Proof. Let $A := J^{pq}(\mathbb{Q})_{\text{tor}}$ and $A_\ell := A \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. Then A_ℓ is a \mathbb{T}_ℓ -module. By Eichler-Shimura relation and the isogeny $\Psi(pq)$, which is Hecke-equivariant, for a prime r not dividing pq

$$T_r \equiv \text{Frob}_r + \text{Ver}_r \text{ on } J_{/\mathbb{F}_r}^{pq},$$

where Frob_r is the Frobenius morphism in characteristic r and Ver_r is its transpose. Therefore $T_r - 1 - r$ kills A and hence A_ℓ is annihilated by I_0 , i.e., A_ℓ is a \mathbb{T}_ℓ/I_0 -module. By Proposition 2.3, it decomposes into A_ℓ^i , where each A_ℓ^i is a \mathbb{T}_ℓ/I_i -module. More precisely, we have $A_\ell^i = A_\ell \cap J^{pq}[I_i] = A_\ell[I_i]$. Thus, it suffices to prove that $A_\ell^i = 0$ for all $1 \leq i \leq 4$.

If all the above assumption do not hold, then \mathfrak{m}_1 is not maximal. Therefore $A_\ell^1 = 0$. By Theorem 3.3, we have $A_\ell^2 = 0$ as well. Now we assume that $A_\ell^3 \neq 0$. If $\ell = 3$, then we further assume that ℓ does not divide $(p-1)(q-1)$. Then $A_\ell^3[\ell] \simeq (\mathbb{Z}/\ell\mathbb{Z})^a$ for some $a \geq 1$. Since $A_\ell^3[\ell] = A_\ell[\ell, I_3] = A_\ell[\mathfrak{m}_3]$, we have $\mathbb{Z}/\ell\mathbb{Z} \subseteq J^{pq}[\mathfrak{m}_3]$. This contradicts Theorem 4.1. Thus, we have $A_\ell^3 = 0$ and hence $A_\ell^4 = 0$ by symmetry. \square

6. THE KERNEL OF AN ISOGENY DUE TO RIBET

In this section, we provide an application of our main theorem. As before, let $J_0(pq)^{\text{new}}$ denote the new quotient of $J_0(pq)$, $\Psi(pq)$ denote an isogeny from $J_0(pq)^{\text{new}}$ to J^{pq} , and let $K(pq)$ denote the kernel of $\Psi(pq)$:

$$0 \longrightarrow J_0(pq)_{\text{old}} \longrightarrow J_0(pq) \xrightarrow{\pi} J_0(pq)^{\text{new}} \longrightarrow 0;$$

$$0 \longrightarrow K(pq) \longrightarrow J_0(pq)^{\text{new}} \xrightarrow{\Psi(pq)} J^{pq} \longrightarrow 0.$$

Ogg [12] conjectured that the image of some cuspidal divisors in $J_0(pq)$ is contained in $K(pq)$. This conjecture is proved by González and Molina [7] if the genus of \mathcal{X}^{pq} is at most 3. We prove some of the conjecture by Ogg as follows:

Theorem 6.1. *Let ℓ^m and ℓ^n be the exact powers of ℓ dividing $p+1$ and $q+1$, respectively. If $\ell \geq 5$ and all the following conditions hold, then $K(pq)$ contains $\pi(\mathcal{C}_\ell(pq))$, and the latter is isomorphic to $\mathbb{Z}/\ell^m\mathbb{Z} \oplus \mathbb{Z}/\ell^n\mathbb{Z}$:*

- ℓ does not divide $(p-1, q-1)$;
- if $p \equiv 1 \pmod{\ell}$, then $q^{\frac{p-1}{\ell}} \not\equiv 1 \pmod{p}$;
- if $q \equiv 1 \pmod{\ell}$, then $p^{\frac{q-1}{\ell}} \not\equiv 1 \pmod{q}$.

If $\ell = 3$ and $(p-1)(q-1)$ is not divisible by 3, then $K(pq)$ contains $\pi(\mathcal{C}_3(pq))$, and the latter is isomorphic to $\mathbb{Z}/3^\alpha\mathbb{Z} \oplus \mathbb{Z}/3^\beta\mathbb{Z}$, where $\alpha = \max\{0, m-1\}$ and $\beta = \max\{0, n-1\}$.

Proof. Let $C_p := [P_1 - P_p]$ and $C_q := [P_1 - P_q]$ be elements in $\mathcal{C}(pq)$, where P_t is the cusp of $X_0(pq)$ corresponding to $1/t \in \mathbb{P}^1(\mathbb{Q})$.

Assume that $\ell \geq 5$. Let $(p-1)(q^2-1) = \ell^a \times x$ and $(q-1)(p^2-1) = \ell^b \times y$, where ℓ does not divide xy . Let $D_p := xC_p$ and $D_q := yC_q$. Assume that all the above three conditions hold. Then by Chua-Ling [3], we have $\mathcal{C}_\ell(pq) \simeq \langle D_p \rangle \oplus \langle D_q \rangle$ and it is contained in $J_0(pq)(\mathbb{Q})_{\text{tor}}$. By symmetry, we may assume that $q \not\equiv 1 \pmod{\ell}$. Then, the intersection of $\mathcal{C}_\ell(pq)$ and $J_0(pq)_{\text{old}}$ is isomorphic to $\langle \ell^n D_p \rangle \oplus \langle \ell^m D_q \rangle$ (cf. [3, Theorem 2]). Thus, $\pi(\mathcal{C}_\ell(pq)) \simeq \mathbb{Z}/\ell^n\mathbb{Z} \oplus \mathbb{Z}/\ell^m\mathbb{Z}$. Since $J^{pq}(\mathbb{Q})_{\text{tor}, \ell} = 0$ by Theorem 1.2, $K(pq)$ contains $\pi(\mathcal{C}_\ell(pq))$.

Assume that $\ell = 3$ and 3 does not divide $(p-1)(q-1)$. Note that the order of C_p (resp C_q) is the numerator of $\frac{(p-1)(q^2-1)}{3}$ (resp. $\frac{(q-1)(p^2-1)}{3}$) up to powers of 2. Thus, $\mathcal{C}_3(pq)$ is isomorphic to $\mathbb{Z}/3^\alpha\mathbb{Z} \oplus \mathbb{Z}/3^\beta\mathbb{Z}$. Since the 3-primary subgroups of the rational torsion subgroups of $J_0(pq)_{\text{old}}$ and J^{pq} are zero, $K(pq)$ contains $\pi(\mathcal{C}_3(pq))$, and the latter is isomorphic to $\mathbb{Z}/3^\alpha\mathbb{Z} \oplus \mathbb{Z}/3^\beta\mathbb{Z}$. \square

Remark 6.2. Let p and q be distinct primes with $p < q$ and let S be the set of pairs (p, q) such that $g(\mathcal{X}^{pq}) \leq 3$. In this case, González and Molina determined the kernel of $K(pq)$ by taking some precise isogeny between $J_0(pq)^{\text{new}}$ and J^{pq} . Let S_ℓ be the subset of S consisting of the pairs satisfying all the above three conditions with respect to ℓ . Then, the following table describes the orders of $K(pq)$ (for their chosen $\Psi(pq)$) and

$$\mathcal{D}(pq) := \bigoplus_{\substack{\ell \text{ odd primes} \\ \text{such that } (p, q) \in S_\ell}} \pi(\mathcal{C}_\ell(pq)).$$

If ℓ is large enough, then $\mathcal{C}_\ell(pq) = 0$ and hence the direct sum in the definition is actually a finite sum. Moreover, from its definition and the above theorem, $\mathcal{D}(pq) \subseteq \pi(\mathcal{C}(pq)) \cap K(pq)$. We can see that $K(pq)/\mathcal{D}(pq)$ is a 2-group for any $(p, q) \in S$ from the table below.

S	$g(\mathcal{X}^{pq})$	$\in S_3?$	$\in S_5?$	$\in S_7?$	$\#\mathcal{D}(pq)$	$\#K(pq)$
(2, 7)	1	No	Yes	Yes	1	2
(2, 17)	1	Yes	Yes	Yes	3	3
(3, 5)	1	Yes	Yes	Yes	1	1
(3, 7)	1	No	Yes	Yes	1	2
(3, 11)	1	Yes	Yes	Yes	1	1
(2, 13)	2	No	Yes	Yes	7	7
(2, 19)	2	No	Yes	Yes	5	5
(2, 29)	2	Yes	Yes	Yes	5	5
(2, 31)	3	No	Yes	Yes	1	8
(2, 41)	3	Yes	Yes	Yes	7	7
(2, 47)	3	Yes	Yes	Yes	1	4
(3, 13)	3	No	Yes	Yes	7	7
(3, 17)	3	Yes	Yes	Yes	3	3
(3, 19)	3	No	Yes	Yes	5	20
(3, 23)	3	Yes	Yes	Yes	1	8
(5, 7)	3	No	Yes	Yes	1	2
(5, 11)	3	Yes	Yes	Yes	1	1

Table 1.

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